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CONTINUAL FRACTURE OF NON-LINEARLY ELASTIC BODIES*

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A model of a medium is considered for which the development of microdefects, namely, microcracks, pores and intergrain vacancies, are substantial. Microdefect diffusion and polar phenomena are neglected. The principles of the phenomenological description of the behaviour of such materials are found in damage theories /1, 2/. The relation between the strain history and the strength criterion of a material particle is revealed /3/. The energy aspects of the damage phenomenon were considered /4-7/ in the modelling of fracture waves by strong discontinuities separating materials with different rheological properties. A survey of the further development of the damage theory and its applications to viscoelasticity and creep problems is contained in /8/.

The approach used in this paper differs from the traditional approach in two main ways. Firstly, by the axiom of local energy balance and the inequality of dissipation. Following the mechanics of an isolated crack /9/, a term describing the energy flux distributed over a volume that is related to transformation of part of the stored energy and the work of the stresses in the surface energy of the microcracks, is introduced into the local energy balance equation. The dissipation inequality differs from the Clausius-Duhem inequality /10/ by a component that is the dissipation of fracture. The second essential difference is the modification of the principle of macroscopic determinability. It is considered that the running reaction of a material element is a functional determined by the independent prehistories of the strain, entropy, and damage.

For a thermoelastic damaged body these fundamental assumptions enable one to clarify two important questions that remain unsolved in the traditional theory of damage: what is the macroscopic interpretation of the measure of material damage and among what group of equations is the equation giving the evolution of damage, the group of rheological relationships or the conservation laws? It is shown that for the medium under consideration, the second rank damage tensor can be identified with the macroscopic strains of an element unloaded from the running state in a passive process. It is found that damage growth in thermoelastic media is not controlled by the kinetic equation but by a finite relationship connecting the running value of the damage with the running value of the strain, entropy, and the distributed source of the damage. This relationship is a consequence of two thermodynamics principles, i.e., it belongs to the transformed conservation laws.

It is shown that within the framework of the assumptions taken for initially isotropic bodies, two and only two kinds of materials are possible whose damage is characterized by either a scalar or a symmetric second-rank tensor. Conditions are formulated which the governing relationships should satisfy so that the dissipation inequality and the invariance requirements would be satisfied in all the allowable processes. The conditions presented significantly narrow down the class of allowable equations of state, especially for a material with a scalar damage characteristic.

Assuming small strains and that the rheology is independent of the temperature, the simplest example is constructed of a medium simulating a number of qualitative effects typical for mountain rock /11/.

1. Kinematics and laws of conservation. Let $\mathbf{X}$ be the radius-vector of a material particle of a body in the initial configuration $\varkappa$, $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ is the radius-vector of the same particle at the time $t > 0$ corresponding to a running configuration χ. Let $\mathbf{F}$ denote the

gradient of the non-degenerate mapping $\varkappa \to \chi$, such that $d\mathbf{x} = \mathbf{F}d\mathbf{X}$, $\det \mathbf{F} \neq 0$. We will consider that in addition to the displacement vector $\mathbf{u}(\mathbf{X}, t) = \mathbf{x} - \mathbf{X}$ and the absolute temperature $\theta(\mathbf{X}, t) > 0$, a certain tensor π of the second rank is defined on $\varkappa$ and is a measure of material damage. Unlike in /1-3/, it is assumed that the tensor $\pi(t)$ is neither determined by the running value $\{\mathbf{F}(t), \theta(t)\}$ nor even by the whole prehistory $\{\mathbf{F}(\tau), \theta(\tau)\}, -\infty < \tau \leqslant t$ in the general case.

Here the situation is analogous to the theory of moment (polar) media /12/ in which the rotation vector and its gradient are considered in addition to the displacement vector and its gradient for the description of additional degrees of freedom of the macroparticles. In the general case rotation is not determined by displacement.

The assumption of the possibility of a change in the damage tensor π independently of the strain and temperature histories indicates the qualitative distinction between π and the plastic or viscous strains tensor, which is a parameter reflecting the influence of the prehistory on the running state /13/.

As an illustration we present the tensor

$$\pi = \frac{1}{V}\int_S \frac{1}{2}(\mathbf{u}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{u})\,dS$$

used /14, 15/ as a measure of material damage. Here V is the volume of material containing weakly opened cracks, whose middle surface element dS is characterized by the normal $\mathbf{n}$ and the displacement vector $\mathbf{u}$ of the microcrack edges, and $\otimes$ is the symbol of dyad multiplication. Obviously both the normal and the displacement vector are characteristics of the microstructure and can change under the action of factors of non-thermomechanical nature for a fixed macroscopic strain and temperature.

Let us now formulate the conservation laws for a medium being damaged. Let $\rho_\varkappa, \rho$ be the mass density at $\varkappa$ and χ respectively, $\mathbf{b}$ the mass force vector, and $\mathbf{T}_\varkappa$ the non-symmetric Piola-Kirchhoff stress tensor of the first kind. The laws of conservation of mass, momentum, and anuglar **momentum** and the compatibility of the velocities and strains are written exactly as in the case of a continuous medium in which there are no damage evolution processes

$$\begin{aligned} &\rho_\varkappa \partial\mathbf{v}/\partial t\,|_X - \operatorname{Div}\mathbf{T}_\varkappa = \rho_\varkappa \mathbf{b}, \quad \mathbf{T}_\varkappa \mathbf{F}^T = \mathbf{F}\mathbf{T}_\varkappa^T \\ &\partial\mathbf{F}/\partial t\,|_X - \operatorname{Div}(\mathbf{v}\otimes\mathbf{I}) = 0, \quad \rho \det \mathbf{F} = \rho_\varkappa \end{aligned} \tag{1.1}$$

(Div is the divergence in the variables $\mathbf{X}$, and $\mathbf{I}$ is the unit tensor of second rank).

The local energy balance equation is taken in the form /16, 17/

$$\begin{aligned} &\rho_\varkappa \varepsilon^\cdot = \mathbf{T}_\varkappa : \mathbf{F}^\cdot + Q_T + Q_f \\ &Q_T = \mathbf{Div}\,\mathbf{q}_\varkappa + \rho_\varkappa r_T, \quad Q_f = \rho_\varkappa (r_f - \varepsilon_f^\cdot) \end{aligned} \tag{1.2}$$

Here $\mathbf{T}_\varkappa : \mathbf{F}^\cdot$ is the work intensity of the stress in a momentless medium (the colon denotes convolution in the two indices), Q_T is the energy influx due to thermal conduction ($\mathbf{q}_\varkappa$ is the heat flux vector in the variables $\mathbf{X}$) and the action of the distributed heat sources r_T. The energy influx Q_f is due to distributed sources and sinks of a non-thermomechanical nature. The scalar quantity r_f is the density of the distributed energy sources associated with the change in material damage because of external action on the microdefect geometry and quantity. It is an arbitrarily given external field, and in particular, can equal zero identically. Unlike r_f the quantity $\varepsilon_f^\cdot$ is non-zero in any damage change process and is the density of the distributed surface energy. The connection between the damage phenomenon and the change in the free surface of the microcrack edges is emphasised by the introduction of $\varepsilon_f^\cdot$.

It should be noted that the Griffith energy relationship in the mechanics of an isolated crack /9, 18/ rapidly follows from (1.2) if r_f and $\varepsilon_f^\cdot$ (apart from a coefficient) are δ-functions whose supports agree with the moving vertex of the isolated crack.

Taking (1.1) into account, (1.2) can be converted into a local energy conservation law

$$\rho_\varkappa \frac{\partial}{\partial t}\left(\varepsilon + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + \varepsilon_f\right) - \operatorname{Div}(\mathbf{T}_\varkappa^T\mathbf{v} + \mathbf{q}_\varkappa) = \rho_\varkappa(\mathbf{b}\cdot\mathbf{v} + r_T + r_f)$$

Let η be the entropy density, that is the conjugate reaction of the material to the introduction of the temperature θ into the number of state parameters of a material particle. Let $\mathbf{\Pi}$ denote the material reaction to the damage tensor π. Then the second principle of thermodynamics can be formulated in the form of a local inequality ($\nabla_\varkappa$ is the gradient in the variable $\mathbf{X}$)

$$\delta_M + \delta_T + \delta_f \geqslant 0 \tag{1.3}$$

$$\begin{aligned} &\delta_M \equiv \theta\eta^\cdot - Q_T = \theta\eta^\cdot - (\rho_\varkappa^{-1}\operatorname{Div}\mathbf{q}_\varkappa + r_T) \\ &\delta_T \equiv (\rho_\varkappa\theta)^{-1}\mathbf{q}_\varkappa\nabla_\varkappa\theta \\ &\delta_f \equiv \pi : \mathbf{\Pi}^\cdot - Q_f = \pi : \mathbf{\Pi}^\cdot + \varepsilon_f^\cdot - r_f \end{aligned}$$

Here δ_M is the internal (mechanical) dissipation that figures in the Planck inequality of classical thermomechanics /10/, δ_T is the thermal dissipation associated with the thermal conduction of the medium and occurring in the Fourier inequality /10/, while δ_f is the dissipation of the continual fracture /16, 17/.

It is seen from a comparison of the expressions for δ_M and δ_f that $\mathbf{\Pi}$ is the tensor analogue of the entropy density. The difference between the energy influxes Q_T and Q_f is explained by neglecting the microdefect diffusion in the model under consideration.

If the local energy balance Eq.(1.2) from which $Q_T + Q_f = \rho_\varkappa \varepsilon^\cdot - \mathbf{T}_\varkappa : \mathbf{F}^\cdot$ follows, is used we arrive at a form of the dissipation inequality

$$-\varepsilon^\cdot + \rho_\varkappa^{-1}\mathbf{T}_\varkappa : \mathbf{F}^\cdot + \theta\eta^\cdot + \pi : \mathbf{\Pi}^\cdot + \delta_T \geqslant 0 \tag{1.4}$$

convenient for studying the constraints imposed by the second law of thermodynamics on the governing relations.

2. Governing relationships of a thermoelastic material with damage. The introduction of fields associated with surface phenomena and having a non-thermomechanical nature in the classical sense, results in the need to modify the fundamental principle of the theory of the governing equations, the principle of macroscopic definability /13/. Using the notation

$$\lambda(\mathbf{X}, \tau) \equiv \{\mathbf{F}(\mathbf{X}, \tau), \eta(\mathbf{X}, \tau), \mathbf{\Pi}(\mathbf{X}, \tau), \nabla_\varkappa\theta(\mathbf{X}, \tau)\}, \quad \tau \leqslant t$$
$$\Sigma(\mathbf{X}, t) \equiv \{\varepsilon(\mathbf{X}, t), \mathbf{T}_\varkappa(\mathbf{X}, t), \theta(\mathbf{X}, t), \pi(\mathbf{X}, t), e_f(\mathbf{X}, t), \mathbf{q}_\varkappa(\mathbf{X}, t)\}$$

the principle of thermodynamically consistent macroscopic definability for media being damaged can be written in the form

$$\Sigma(\mathbf{X}, t) = \Sigma\{\lambda(\mathbf{X}, \tau), \mathbf{X}, t\}_{\tau=-\infty}^{\tau=t} \tag{2.1}$$

where the governing functionals $\Sigma_{\tau=-\infty}^{\tau=t}$ should satisfy the dissipation inequality (1.4) for any prehistories $\lambda(\tau)$, if the derivatives $\lambda^\cdot$ and $\varepsilon^\cdot$ exist at the time $\tau = t$.

Homogeneous non-ageing materials are examined later, in which connection there is no explicit dependence of the right-hand sides of (2.1) on $\mathbf{X}, t$ and these arguments are omitted everywhere.

We define a non-linear thermoelastic material being damaged as a medium for which the mappings (2.1) reduce to functions of the running values

$$\Sigma(t) = \Sigma^+\{\lambda(t)\} \tag{2.2}$$

Substituting (2.1) into the inequality (1.4) and using standard reasoning /10/ associated with the construction of a local linear continuation of the process $\lambda(\tau)$, that is possible because of the independence of the rate of change $\lambda^\cdot(\tau)$, we obtain the necessary and sufficient conditions for satisfying inequality (1.4)

$$\mathbf{T}_\varkappa = \rho_\varkappa \partial\varepsilon/\partial\mathbf{F}, \quad \theta = \partial\varepsilon/\partial\eta, \quad \partial\varepsilon/\partial(\nabla_\varkappa\theta) = 0, \quad \pi = \partial\varepsilon/\partial\mathbf{\Pi} \tag{2.3}$$
$$\delta_T \geqslant 0, \quad \delta_M + \delta_f = 0 \tag{2.4}$$

The first three relationships (2.3) agree with the relationships of classical non-linear thermoelasticity /10/. The last formula of (2.3), justifying to some extent the terminology "fracture entropy" for the tensor $\mathbf{\Pi}$, enables us to give a graphical interpretation to the fracture dissipation δ_f for $r_f = 0$. Indeed, in this case $\delta_f = (\partial\varepsilon/\partial\mathbf{\Pi}) : \mathbf{\Pi}^\cdot + e_f^\cdot$, i.e., the dissipation of continual fracture equals the difference between the rate of absorption energy in the formation of the new free surface and the rate of energy dissipation because of the damage growth and the formation of new free microcrack edges.

Later we shall assume that the distributed energy sink $e_f^\cdot$, togehter with the sources r_f, equal zero if $\mathbf{\Pi}^\cdot = 0$. Since

$$e_f^\cdot = \frac{\partial e_f}{\partial \mathbf{F}} : \mathbf{F}^\cdot + \frac{\partial e_f}{\partial \eta}\eta^\cdot + \frac{\partial e_f}{\partial \gamma_\varkappa}\gamma_\varkappa^\cdot = 0 \quad (\gamma_\varkappa \equiv \nabla_\varkappa\theta)$$

in such processes, then by virtue of the independence of $\mathbf{F}^\cdot$, $\eta^\cdot$, and $\gamma_\varkappa^\cdot$ the derivatives $\partial e_f/\partial\mathbf{F}$, $\partial e_f/\partial\eta$, $\partial e_f/\partial\gamma_\varkappa$ equal zero identically so that

$$e_f = e_f(\mathbf{\Pi}), \quad e_f^\cdot = \mathbf{G} : \mathbf{\Pi}^\cdot, \quad \mathbf{G} \equiv \partial e_f(\mathbf{\Pi})/\partial\mathbf{\Pi} \tag{2.5}$$

The quantity $\mathbf{G}$ plays the part of the tensor of material resistance of fracture.

To satisfy the condition $r_f = 0$ for $\mathbf{\Pi}^\cdot \neq 0$, it is sufficient that

$$r_f = \mathbf{G}_* : \mathbf{\Pi}^\cdot \tag{2.6}$$

where $\mathbf{G}_*$ is an arbitrary given tensor of the second rank governing the density of the external

fracture sources.

Since we have $\delta_f = 0$ for $\Pi^\cdot = 0$ when taking account of (2.5) and (2.6), then there follows from the second relationship of (2.4) and the fact that δ_M is independent of $\Pi^\cdot$ $\delta_M = 0, \delta_f = 0$ for all $\Pi^\cdot$. This means that the thermoelastic, damageable material is perfect both in the thermomechanics sense as well as in the fracture energy sense. The relationship $\delta_M = 0$ shows that any process proceeds without internal dissipation so that the rate of change of the internal energy for fixed strain and damage is exactly equal to the thermal influx Q_T. The equality $\delta_f = 0$ means that the rate of absorption of the energy expended in changing ε_f equals the rate of energy supply because of the distributed fracture sources r_f and the elastic energy release during damage growth.

The relationship $\delta_M = 0$ results in the equation

$$\rho_\varkappa \eta^\cdot - \mathrm{Dif}\,(\theta^{-1}\mathbf{q}_\varkappa) = \rho_\varkappa(\theta^{-1}r_T + \delta_T)$$

which is the law of conservation (of the change) of entropy in domains of the smooth solutions.

Taking (2.6) into account, relationship $\delta_f = 0$ yields

$$(\partial\varepsilon/\partial\Pi + \mathbf{G} - \mathbf{G}_*) : \Pi^\cdot = 0 \tag{2.7}$$

Hence, by virtue of the arbitrariness of $\mathbf{G}_*$ it follows that two processes are possible: a passive one in which $\Pi^\cdot = 0$, i.e., the damage is invariant, and an active one in which

$$\partial\varepsilon(\mathbf{F}, \eta, \Pi)/\partial\Pi + \mathbf{G}(\Pi) - \mathbf{G}_*(t) = 0, \quad \Pi^\cdot \neq 0 \tag{2.8}$$

Eq.(2.8) determines the relation between the damage entropy Π and the running value of the strain $\mathbf{F}$, the entropy η, and the tensor $\mathbf{G}_*$ of the distributed fracture sources. Differentiating (2.8) with respect to time, we find the relation between $\Pi^\cdot$ and the velocities $\mathbf{F}^\cdot, \eta^\cdot$, and $\mathbf{G}_*^\cdot$:

$$\left(\frac{\partial^2\varepsilon}{\partial\Pi\partial\Pi} + \frac{\partial\mathbf{G}}{\partial\Pi}\right) : \Pi^\cdot = \mathbf{G}_*^\cdot - \left(\frac{\partial^2\varepsilon}{\partial\Pi\partial\mathbf{F}} : \mathbf{F}^\cdot + \frac{\partial^2\varepsilon}{\partial\Pi\partial\eta}\eta^\cdot\right) \tag{2.9}$$

Furthermore, we will consider materials in which microdefect healing processes can be neglected. The surface energy density stands out as a natural characteristic of the degree of material fracturability (the norms of the damage entropy tensor). Then the active loading condition is

$$\varepsilon_f^\cdot = \mathbf{G} : \Pi^\cdot > 0 \tag{2.10}$$

Together with (2.8) condition (2.10) imposes a constraint on the state $(\mathbf{F}, \eta, \Pi)$ and the rates of change $(\mathbf{F}^\cdot, \eta^\cdot, \mathbf{G}_*^\cdot)$ for which the active process is realized.

The possibility of passive continuation $(\Pi^\cdot = 0)$ of the process from any running state $\{\mathbf{F}(t), \eta(t), \Pi(t)\}$ that is evident from (2.7), means that it is possible to introduce a material element configuration with the very same damage $\Pi(t)$ and entropy $\eta(t)$ but different strain $\mathbf{F}^*(t)$. To isolate the specific properties of this element configuration we will assume its unloaded configuration in which

$$\mathbf{T}_\varkappa\{\mathbf{F}^*(t), \eta(t), \Pi(t)\} = \mathbf{0} \tag{2.11}$$

Let $\varkappa^*$ denote the configuration of a body comprised of unloaded elements. Then the mapping $\varkappa \to \chi$ of the initial into the current configuration can be represented in the form of a sequence of two non-degenerate mappings $\varkappa \to \varkappa^*, \varkappa^* \to \chi$ with the gradients $\mathbf{F}^*$ and $\mathbf{E}$ so that we have the composition

$$\mathbf{F} = \mathbf{E}\mathbf{F}^*, \ \det\mathbf{E} \neq 0, \ \det\mathbf{F}^* \neq 0 \tag{2.12}$$

Under the assumption that (2.11) is solvable uniquely for Π, the tensor $\mathbf{F}^*$ can be considered to be a macroscopic measure of the damage. Understandably this characteristic is not unique to the extent that it is possible to introduce other configurations with the very same damage $\Pi(t)$ but different entropy equal to $\eta_\varkappa = \eta(0)$, say, another state of stress $\mathbf{T}_\varkappa(\mathbf{F}^*, \eta, \Pi) \neq 0$ etc. However, all these measures are equivalent from the viewpoint of the representation of the damage by the strain function. Consequently, we shall henceforth identify Π and $\mathbf{F}^*$.

Unlike $\varkappa$ and χ, the intermediate configuration $\varkappa^*$ does not belong to Euclidean space in the general case, i.e., there is no compatibility equation of the type $(1.1)_3$ for $\mathbf{F}^*$. The concept of a tangential configuration $\varkappa^*(\mathbf{X})$, that is a configuration of a homogeneously strained body with strains equal to the strains at the material point $\mathbf{X}$ /19/, is important for the sequel. This configuration, consisting of identically strained elements (mutually superimposed oblique parallelepipeds) understandably belongs to Euclidean space.

Using the tensor $\Pi \equiv \mathbf{F}^*$ and taking account of (2.3)-(2.5), the governing relationships (2.2) can be written in the form

$$\begin{aligned} &\varepsilon = \varepsilon^\times(\mathbf{F}, \eta, \mathbf{F}^*), \quad \mathbf{T}_\varkappa = \mathbf{T}_\varkappa^\times(\mathbf{F}, \eta, \mathbf{F}^*) \\ &\theta = \theta^\times(\mathbf{F}, \eta, \mathbf{F}^*), \quad \mathbf{q}_\varkappa = \mathbf{q}_\varkappa^\times(\mathbf{F}, \eta, \mathbf{F}^*, \gamma_\varkappa) \end{aligned} \tag{2.13}$$

$$\varepsilon_f = \varepsilon_f^{\varkappa}(\mathbf{F}^*),\ \Phi(\mathbf{F},\ \eta,\ \mathbf{F}^*) = 0$$

where the equation $\Phi = 0$ is the relation between $\mathbf{F}, \eta, \mathbf{F}^*$ under active loading. Relationships (2.13) should satisfy the following invariance requirements.

1°. The principle of material independence from the selection of the reference system /10/;

2°. The condition of invariance relative to orthogonal transformations of any tangential configuration $\varkappa^*(\mathbf{X})$ /19/;

3°. The condition of invariance relative to transformations of the initial configuration $\varkappa$ belonging to the material symmetry group.

We will confine ourselves to the case of initially isotropic media. As in the theory of simple media /10/, this means that for a body fabricated from the material under consideration there is an undistorted or natural configuration $\varkappa_0$, a symmetry group $g_{\varkappa_0}$ which includes the complete orthogonal group o, i.e., $o \subset g_{\varkappa_0}$. Hence, taking account of the maximality of the group o /20/, it follows that two and only two cases are possible:

$$g_{\varkappa_0} = o \tag{2.14}$$

$$g_{\varkappa} = u,\ \forall \varkappa \tag{2.15}$$

where u is a unimodular group of any transformations that do not change volume.

If the symmetry groups (2.14) and (2.15) result in definitions of a solid and a fluid in the theory of simple media, then in the case under consideration both groups are inherent to a solid. This is due to the undamped memory intrinsic to the governing relations (2.13).

A non-linear isotropic thermoelastic damageable material with the symmetry group (2.15) is a material with a scalar damage characteristic. Reasoning in the same way as /19/, we obtain that the following form of the governing relationships (2.13)

$$\begin{aligned}&\varepsilon = \varepsilon^\circ(\mathbf{e}, \eta, \omega),\ \mathbf{T} = \mathbf{T}^\circ(\mathbf{e}, \eta, \omega) = \rho(\mathbf{I} - 2\mathbf{e})\,\partial\varepsilon^\circ/\partial\mathbf{e},\ \theta = \theta^\circ(\mathbf{e}, \eta, \omega) = \\ &\partial e^\circ/\partial\eta,\quad \mathbf{q} = \mathbf{q}^\circ(\mathbf{e}, \eta, \omega, \nabla\theta),\quad \varepsilon_f = \varepsilon_f^\circ(\omega), \\ &\omega = \omega^\circ(\mathbf{e}, \eta)\quad \text{for}\quad \omega^* > 0 \\ &(\mathbf{e} = {}^1/_2(\mathbf{I} - \omega^2\mathbf{F}^{-1T}\mathbf{F}^{-1}))\end{aligned} \tag{2.16}$$

is the necessary and sufficient condition for satisfying the invariance principles 1°-3° and the dissipation inequality (1.4), where $\varepsilon^\circ, \mathbf{T}^\circ, \theta^\circ, \mathbf{q}^\circ, \varepsilon_f^\circ, \omega^\circ$ are isotropic functions of their arguments. The following notation is used in (2.16): $\mathbf{T}$ is a symmetric Cauchy stress tensor, $\mathbf{q}$ is a heat flux vector in the variables $\mathbf{x}$, and $\mathbf{e}$ is an elastic strain tensor analogous in its structure to the Almansi tensor and constructed by using the gradient $\mathbf{E} = \mathbf{F}\Pi^{-1}$ of the mapping $\varkappa^* \to \chi$. It is taken into account here that the symmetric part of the polar expansion Π is the strain due to damage that equals $\omega\mathbf{I}$, $\omega^3 = \det\Pi$.

The other possible symmetry group (2.14) characterizes a material that is not sensitive to orthogonal strains of the undistorted initial configuration. In this case the governing relationships are rather more complex compared with (2.16)

$$\begin{aligned}&\varepsilon = \varepsilon_0(\mathbf{U}, \eta, \omega),\quad \mathbf{T} = \rho\frac{\partial\varepsilon_0}{\partial\mathbf{F}}\mathbf{F}^T,\quad \theta = \frac{\partial\varepsilon_0}{\partial\eta}, \\ &\mathbf{q} = \mathbf{R}\mathbf{q}_0(\mathbf{U}, \eta, \omega, \nabla_\chi\theta),\quad \varepsilon_f = \varepsilon_{f0}(\omega), \\ &\omega = \omega_0(\mathbf{U}, \eta)\qquad \text{for}\qquad \varepsilon_f^* = \mathbf{G} : \omega^* > 0\end{aligned} \tag{2.17}$$

where $\varepsilon_0, \mathbf{q}_0, \varepsilon_{f0}, \omega_0$ are isotropic functions. The following notation is used in (2.17): $\mathbf{R}$ is an orthogonal, and $\mathbf{U}$ is a symmetric positive-definite tensor of the polar expansion $\mathbf{F} = \mathbf{RU}$, and ω is a symmetric positive-definite tensor from the polar expansion $\Pi = \mathbf{R}_\Pi\omega$, $\mathbf{G} \equiv \partial\varepsilon_{f0}(\omega)/\partial\omega$.

3. **Example.** To illustrate the possibilities of the approach utilized, we examine the simplest case. We will assume that the dependence of the material properties on the temperature can be neglected together with the heat flux and the distributed heat sources; the strains are small; the material damage is characterized by a scalar quantity. In addition we assume that $G_* = 0$, $G = \partial\varepsilon_f^\circ/d\omega = \text{const}$.

Let ε be the small strain tensor $\varepsilon' = \varepsilon - {}^1/_3\mathbf{I}(\mathbf{I} : \varepsilon)$ the deviator of the strain tensor, $I_1 = \mathbf{I} : \varepsilon$, $J = (\varepsilon' : \varepsilon')^{1/2}$. We take the potential of an initially isotropic elastic damageable medium in the form

$$\begin{aligned}&\rho_\varkappa\varepsilon = \rho_\varkappa\varepsilon_\varkappa + {}^1/_2KI_1^2 + \mu J^2 + (\gamma - G)\omega + 1/2\beta\omega^2 - \alpha_p I_1\omega - \alpha_s J\omega \\ &(\varepsilon_\varkappa,\ K,\ \mu,\ \gamma,\ \alpha_p,\ \alpha_s,\ \beta = \text{const})\end{aligned} \tag{3.1}$$

A non-linear expression for the stress tensor follows from (3.1)

$$\mathbf{T} = (KI_1 - \alpha_p\omega)\mathbf{I} + (2\mu - \alpha_s\omega/J)\varepsilon' \tag{3.2}$$

The condition $d\varepsilon/d\omega + G = 0$ results for $\omega^{\cdot} > 0$ in the expression for the damage parameter in an active process

$$\omega = \beta^{-1}(\alpha_p I_1 + \alpha_s J - \gamma) \tag{3.3}$$

It is seen from (3.2) that when there is no damage ($\omega = 0$), the coefficients K and μ are the volume modulus and shear modulus of the material.

In the one-dimensional strain case when $\varepsilon_{11} = \varepsilon$, while the remaining strain components equal zero, it follows from (3.2) and (3.3) that

$$T_{11} = \begin{cases} (\lambda + 2\mu)\,\varepsilon \\ (\lambda + 2\mu - \beta\alpha_+^2)\,\varepsilon + \sigma_0^+ \\ (\lambda + 2\mu - \beta\alpha_-^2)\,\varepsilon + \sigma_0^- \end{cases}$$

$$\omega = \begin{cases} 0, & \varepsilon_0^- \leqslant \varepsilon \leqslant \varepsilon_0^+ \\ \alpha_+(\varepsilon - \varepsilon_0^+), & \varepsilon > \varepsilon_0^+ > 0, \quad \varepsilon^{\cdot} > 0 \\ \alpha_-(\varepsilon - \varepsilon_0^-), & \varepsilon < \varepsilon_0^- < 0, \quad \varepsilon^{\cdot} < 0 \end{cases}$$

$$\alpha_\pm = \beta^{-1}(\alpha_p \pm \alpha_s\sqrt{2/3}), \quad \varepsilon_0^\pm = \gamma/\alpha_\pm$$

$$\lambda = K - 2\mu/3, \quad \sigma_0^\pm = \beta\alpha_\pm^2\varepsilon_0^\pm$$

A graph illustrating the dependence mentioned is shown in Fig.1. In the domain boundary by ε_0^+ under tension and ε_0^- under compression, the material behaves as a linearly elastic body with Lamé coefficients λ and μ. Outside this domain the dependence on the strain is also linear for $\varepsilon\varepsilon^{\cdot} > 0$ but with the tangential moduli $\lambda + 2\mu - \beta\alpha_\pm^2$ (the plus sign corresponds to tension and the minus to compression). If the rate of strain at the point $\varepsilon_*^\pm$ changes sign, a passive unloading process starts in which the damage is invariant and equal to $\omega = \alpha_\pm(\varepsilon_*^\pm - \varepsilon_0^\pm)$. The tangential moduli during the unloading exactly equal the elastic moduli of the undamaged material.

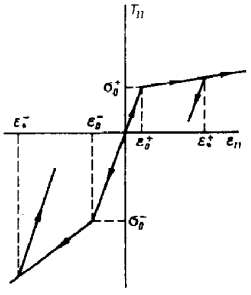

Fig.1

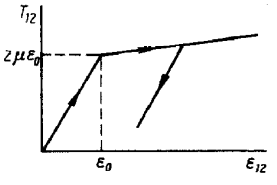

Fig.2

In the case of pure hydrostatics when $\varepsilon = 1/3 I_1 \mathbf{I}$, we have

$$\mathbf{T} = \begin{cases} K I_1 \mathbf{I} \\ [K I_1 - \beta^{-1}\alpha_p^2 (I_1 - I_1^0)]\,\mathbf{I} \end{cases}, \quad \omega = \begin{cases} 0, & I_1 < I_1^0 \\ \beta^{-1}\alpha_p (I_1 - I_1^0), & I_1 > I_1^0, \quad I_1^{\cdot} > 0 \end{cases}$$

where $I_1^0 = \gamma/\alpha_p$. Therefore, hydrostatic compression cannot cause fracture of the material being considered.

Under shear, when $\varepsilon_{12} = \varepsilon_{21} = \varepsilon$, while all the remaining strain tensor components equal zero, fracture starts for $\varepsilon = \varepsilon_0 = \gamma/(\alpha_s\sqrt{2})$ and the stress $T_{12} = 2\mu\varepsilon_0$. The symmetric dependence relative to the origin $T_{12}(\varepsilon_{12})$ is represented in Eig.2. In an active loading process ($\varepsilon > \varepsilon_0$, $\varepsilon^{\cdot} > 0$)

$$\omega = \beta^{-1}\alpha_s\sqrt{2}\,(\varepsilon - \varepsilon_0), \quad T_{11} = -\beta^{-1}\alpha_p\alpha_s\sqrt{2}\,(\varepsilon - \varepsilon_0), \quad T_{12} = (2\mu - \beta^{-1}\alpha_s^2)\varepsilon + \beta^{-1}\alpha_s^2\varepsilon_0$$

i.e., the simple shear strain of the material under consideration is accompanied by the appearance of normal stresses.

Similarly, shear of the material by a tangential stress results in the appearance of dilatancy effects accompanying the fracture process.

Therefore, the model under consideration for an elastic damageable medium describes a number of qualitative effects typical for mountain rocks /11/: the presence of threshold stresses for which fracture starts, the absence of irreversible strains for purely hydrostatic compression, the difference between shear fracture and separation fracture that is characterized by the coefficients α_s and α_p, the material dilatancy, and the elastic nature of the unloading.

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